Propagators with the Mandelstam-Leibbrandt Prescription in the Light-Cone Gauge

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We show that the Feynman propagator in the light-cone gauge with the Mandelstam-Leibbrandt prescription has a logarithmic growth for large $\tilde{n}\cdot x$ which is related to the presence of a residual gauge invariance. Furthermore, we show that the retarded propagator for the $\tilde{n}\cdot A$ component of the gauge field develops a coordinate dependent mass. We argue that this feature is unphysical and may be eliminated by fixing the residual gauge degrees of freedom.

PACS numbers: 11.15.-q, 11.10.Ef, 12.38.Lg

arXiv:hep-th/0412071v2 28 Mar 2005

I. INTRODUCTION

In the study of gauge theories, the light-cone gauge has proven to be quite intriguing and challenging [1]. We recall that in a general axial type gauge (where $n^2\neq 0$ in general and which includes both temporal as well as axial gauges)

$$n\cdot A=0, \tag{1}$$

the path integral propagator in the momentum space takes the form

$$D^{(\mathrm{PI})}_{\mu\nu}(n,p)=-\frac{i}{p^2+i\epsilon}\left[\eta_{\mu\nu}-\frac{n_\mu p_\nu+n_\nu p_\mu}{(n\cdot p)}+\frac{n^2}{(n\cdot p)^2}\,p_\mu p_\nu\right]. \tag{2}$$

(In the case of non-Abelian theories, the gauge potential will correspond to a matrix in the adjoint representation while the propagator will have an identity matrix multiplying the expression (2).) The additional poles of the form $\frac{1}{(n\cdot p)}$ in (2) require a prescription for the propagator to be well defined. In a recent paper [2], we showed that this arbitrariness in the propagator is a consequence of a residual gauge symmetry in the generating functional and fixing this determines the propagator uniquely. The explicit form of the propagator determined in [2] is well behaved for large values of $n\cdot x$. In contrast, in the light-cone gauge where $n^2=0$, we showed that the prescription dependence persists even after a residual gauge fixing is used. We traced the origin of this arbitrariness to the presence of an additional local symmetry in the generating functional.

We note that the Mandelstam prescription [3] (the Leibbrandt prescription [4] is equivalent to that of Mandelstam, but we use the Mandelstam prescription explicitly through out our discussions) is widely used in the light-cone gauge calculations. We recall that this prescription corresponds to defining the additional pole as

$$\frac{1}{[(n\cdot p)]}=\lim_{\epsilon\rightarrow 0}\ \frac{1}{n\cdot p+i\epsilon\tilde{n}\cdot p}. \tag{3}$$

Here $\tilde{n}^\mu$ is a second light-like vector such that $n\cdot\tilde{n}\neq 0$. Such a prescription allows for a Wick rotation of the propagator to Euclidean space [1, 5]. Although calculations with the Mandelstam prescription lead to the correct behavior of physical quantities such as the Wilson line [6, 7, 8], it is also known that this prescription gives rise to some unphysical features such as nonlocal ultraviolet divergent terms in loop diagrams [1]. We would, therefore, like to study systematically the structure of propagators in the light-cone gauge using the Mandelstam prescription. This leads to some interesting and surprising results including the fact that this prescription induces an unphysical behavior even at the tree level. More specifically, we find that this prescription leads to a propagator which has a logarithmic growth for large values of $\tilde{n}\cdot x$. Following our earlier argument, this would correspond to the fact that the Mandelstam prescription does not completely fix the light-cone gauge and we determine this residual gauge symmetry explicitly. The second result that comes out of our analysis is that the light-cone gauge and the Mandelstam prescription induce a coordinate dependent mass for the $\tilde{n}\cdot A$ component of the gauge field which can be seen from the analytic structure of the retarded propagator as well as from the spectral function. This unexpected behavior turns out to be a consequence of the presence of the residual gauge symmetry in the theory.

Our paper is organized as follows. In section **2**, we study the Feynman propagator in the coordinate space with the Mandelstam prescription. We show that the propagator is not well behaved for large values of $\tilde{n}\cdot x$. Following our earlier observations [2], we show that this is a reflection of the fact that the Mandelstam prescription does not fix the light-cone gauge completely and that the free theory has a residual local symmetry. In section **3**, we study in some

detail the structure of the retarded propagator, which indicates that in the light-cone gauge with the Mandelstam prescription, a coordinate dependent mass is induced for the $\tilde{n} \cdot A$ mode. This conclusion is further supported by the structure of the spectral function. In section **4**, we argue that such an unphysical behavior may be eliminated through an appropriate gauge fixing of the residual gauge degrees of freedom of the theory.

II. FEYNMAN PROPAGATOR IN THE COORDINATE SPACE

Let us begin with some useful notation for carrying out calculations in the light-cone gauge. Let $n^\mu, \tilde{n}^\mu$ represent two light-like vectors such that $n^2 = 0 = \tilde{n}^2$, but $n \cdot \tilde{n} \neq 0$. For simplicity, we will choose sgn $(n \cdot \tilde{n})$ to be positive. Any vector can now be decomposed as

$$V_\mu = \frac{\tilde{n}_\mu}{(n \cdot \tilde{n})}\,(n \cdot V) + \frac{n_\mu}{(n \cdot \tilde{n})}\,(\tilde{n} \cdot V) + V_\mu^{\mathrm{T}}, \tag{4}$$

where

$$n \cdot V^{\mathrm{T}} = 0 = \tilde{n} \cdot V^{\mathrm{T}}. \tag{5}$$

Let us next note that Mandelstam's prescription (3) leads upon Fourier transformation to the Green's function

$$G(x) = \frac{i}{[n \cdot \partial]}\,\delta^4(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\frac{e^{ip\cdot x}}{n \cdot p + i\epsilon \tilde{n} \cdot p} = \frac{1}{2\pi}\,\frac{1}{n \cdot x - i\epsilon \tilde{n} \cdot x}\,\delta^2(x^{\mathrm{T}}). \tag{6}$$

The explicit representation (6) is very useful in evaluating systematically the coordinate representation of any quantity. For example, for a general function $f(p)$, we can write

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\frac{f(p)}{n \cdot p + i\epsilon \tilde{n} \cdot p}\,e^{ip\cdot x} = \int \mathrm{d}^4 x'\,G(x - x') \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,f(p) e^{ip\cdot x'}. \tag{7}$$

In particular, this leads to the following coordinate representation for the Feynman propagator in the light-cone gauge with the Mandelstam prescription

$$\begin{aligned} D_{\mu\nu}^{(\mathrm{PI})}(x) \; =& -\frac{1}{(2\pi)^2}\left[\left(\eta_{\mu\nu} - \frac{n_\mu \tilde{n}_\nu + n_\nu \tilde{n}_\mu}{n \cdot \tilde{n}} + \frac{2(x_\mu^{\mathrm{T}} n_\nu + x_\nu^{\mathrm{T}} n_\mu)(\tilde{n} \cdot x)}{(n \cdot \tilde{n}) x^{\mathrm{T}} \cdot x^{\mathrm{T}}} - \frac{2(\tilde{n} \cdot x) n_\mu n_\nu}{(n \cdot \tilde{n})(n \cdot x)}\right)\frac{1}{x^2 - i\epsilon} \right. \\ & \left. + \frac{n_\mu n_\nu}{(n \cdot x)^2}\,\ln\left(\frac{-x^2 + i\epsilon}{(x^{\mathrm{T}})^2}\right)\right], \end{aligned} \tag{8}$$

where we have identified $(x^{\mathrm{T}})^2 = -x^{\mathrm{T}} \cdot x^{\mathrm{T}} \geq 0$. There are several things to note from the explicit structure of the Feynman propagator (8). We see that, apart from the first two tensor structures, all other terms vanish for $\tilde{n} \cdot x = 0$. This may be understood by noting that, since (6) is formally an integration operator, one may evaluate several terms in the Feynman propagator (2) (with $n^2 = 0$) as

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\frac{e^{ip\cdot x}}{p^2 + i\epsilon}\,\frac{1}{n \cdot p + i\epsilon \tilde{n} \cdot p} = \frac{i}{n \cdot \tilde{n}} \int_{\tilde{n}\cdot x_0}^{\tilde{n}\cdot x} \mathrm{d}(\tilde{n} \cdot x') \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\,\frac{e^{ip\cdot x'}}{p^2 + i\epsilon}, \tag{9}$$

where the "prime" in x' in the exponent refers only to the coordinate $\tilde{n} \cdot x'$. We note that if $\tilde{n} \cdot x = 0$, the exponent in the first expression has no dependence on $(n \cdot p)$ and since the two poles in the integrand of this expression lie on the same side of the complex $(n \cdot p)$ plane, such an integral will give zero in this limit. This implies, from the second expression in (9), that in this case the reference point can be chosen to be the origin [9], which therefore explains the vanishing of the above terms at $\tilde{n} \cdot x = 0$.

On the other hand, for large values of $\tilde{n} \cdot x$, the propagator in (8) has a logarithmic growth and, therefore, it is not well behaved. This mildly singular behavior can be viewed from various points of view. Probably the simplest way is to note that the prescription (3) does not quite regularize the singularities when $(n \cdot p), (\tilde{n} \cdot p) \to 0$ simultaneously. This lack of a bounded behavior of the propagator even with the Mandelstam prescription is consistent with our analysis and following our earlier arguments should correspond to the presence of some residual gauge symmetry. To see, in the simplest way, that the theory has a residual gauge symmetry even with the Mandelstam prescription, let us write an effective Lagrangian density for the free theory which would reproduce the Mandelstam prescription in

the propagator naturally. It is easy to check that the free Lagrangian density (the non-Abelian theory would involve a trace over the matrix indices)

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2}(N \cdot A)\ \frac{\Box^2}{\xi\Box^2 + \epsilon^2(\tilde{n}\cdot\partial)^2}\ (N \cdot A), \tag{10}$$

where

$$N^\mu = n^\mu + i\epsilon\ \frac{(\tilde{n}\cdot\partial)\partial^\mu}{\Box}, \tag{11}$$

leads, in the limit $\xi \to 0$, to the Feynman propagator in the light-cone gauge with the Mandelstam prescription [10]. The fact that this effective Lagrangian density is not Hermitian could signal certain difficulties in a theory incorporating the Mandelstam prescription. However, from the point of view of looking for residual symmetries, the Lagrangian density (10) is suitable for our purpose. We note that under a gauge transformation of the form

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\omega(x), \tag{12}$$

the invariant action will, of course, not change. Furthermore, it follows from the structure in (11) that, under such a transformation

$$N\cdot A \to N\cdot A, \tag{13}$$

if $\omega(x)$ is a function only of $x^{\mu\,\mathrm{T}}$. Thus, the free theory (10) has a residual gauge symmetry with a parameter $\omega(x^{\mathrm{T}})$. As we have argued in our earlier work [2], the presence of this residual symmetry is responsible for the propagator (8) in the light-cone gauge with the Mandelstam prescription having a (mildly) singular behavior for large values of $\tilde{n}\cdot x$.

III. RETARDED PROPAGATOR AND THE SPECTRAL FUNCTION

The Feynman propagator (8) is an analytic function in the upper half of the complex $\tilde{n}\cdot x$ plane except for an isolated pole and a logarithmic branch cut beginning at the light-cone $x^2 = 0$. Therefore, one can easily write down a dispersion relation in the complex $\tilde{n}\cdot x$ plane with one subtraction because of the logarithmic behavior. However, we do not give any further detail on this and instead we now analyze the retarded propagator, which is more useful for a better understanding of the structure of this theory.

The retarded propagator of the theory can be obtained from the form of the Feynman propagator in (8) as [11, 12]

$$\begin{aligned} D_{\mu\nu}^{(\mathrm{PI})\ (\mathrm{R})}(x) &= 2\theta(x^0)\mathrm{Im}(D_{\mu\nu}^{(\mathrm{PI})}) \\ &= -\frac{\theta(x^0)}{2\pi}\left[\left(\eta_{\mu\nu} - \frac{n_\mu\tilde{n}_\nu + n_\nu\tilde{n}_\mu}{(n\cdot\tilde{n})} + \frac{2(x^{\mathrm{T}}_\mu n_\nu + x^{\mathrm{T}}_\nu n_\mu)(\tilde{n}\cdot x)}{(n\cdot\tilde{n})x^{\mathrm{T}}\cdot x^{\mathrm{T}}}\right)\delta(x^2)\right. \\ &\qquad \left. -n_\mu n_\nu\frac{(x^{\mathrm{T}})^2}{(n\cdot x)^2}\left(\delta(x^2) - \frac{1}{(x^{\mathrm{T}})^2}\ \theta(x^2)\right)\right]. \end{aligned} \tag{14}$$

The $\delta(x^2)$ term simply corresponds to a massless pole in the momentum space. However, the coefficient of $n_\mu n_\nu$ shows a more interesting structure of $\theta(x^2)$. To appreciate what this corresponds to, let us define

$$\Delta(x) = \frac{\mathrm{sgn}(x^0)}{2\pi}\ \frac{(n\cdot\tilde{n})(x^{\mathrm{T}})^2}{(n\cdot x)^2}\left(\delta(x^2) - \frac{1}{(x^{\mathrm{T}})^2}\ \theta(x^2)\right), \tag{15}$$

so that we can identify

$$-\frac{\tilde{n}^\mu\tilde{n}^\nu}{n\cdot\tilde{n}}D_{\mu\nu}^{(\mathrm{PI})\ (\mathrm{R})}(x) = -\theta(x^0)\ \Delta = -\frac{\theta(x^0)}{2\pi}\ \frac{(n\cdot\tilde{n})(x^{\mathrm{T}})^2}{(n\cdot x)^2}\left(\delta(x^2) - \frac{1}{(x^{\mathrm{T}})^2}\ \theta(x^2)\right). \tag{16}$$

Comparing this with the retarded propagator for a massive scalar field near the light-cone [11, 12],

$$D^{(\mathrm{R})}(x) = -\frac{\theta(x^0)}{2\pi}\left(\delta(x^2) - \frac{m}{2\sqrt{x^2}}\ J_1(m\sqrt{x^2})\ \theta(x^2)\right) \approx -\frac{\theta(x^0)}{2\pi}\left(\delta(x^2) - \frac{m^2}{4}\ \theta(x^2)\right), \tag{17}$$

we conclude that near the light-cone, the $\tilde{n} \cdot A$ component of the gauge field, in the light-cone gauge with the Mandelstam prescription, has developed a coordinate dependent mass

$$m^2 = \frac{4}{(x^{\mathrm{T}})^2} > 0. \tag{18}$$

We can derive further support for this by looking at the spectral function associated with the $\tilde{n} \cdot A$ component of the gauge field. Let us recall that the spectral function [11, 12] may be defined as

$$\frac{i}{n \cdot \tilde{n}} \left[\tilde{n} \cdot A(x), \tilde{n} \cdot A(y)\right] = \Delta(x-y) = \frac{(n \cdot \tilde{n})(z^{\mathrm{T}})^2}{(n \cdot z)^2}\, i \int \frac{\mathrm{d}^4 p}{(2\pi)^3} \left(e^{-ip\cdot z} - e^{ip \cdot z}\right) \rho, \tag{19}$$

where for simplicity of notation we have identified $z^\mu = (x^\mu - y^\mu)$. From the structure of (15), we note that the covariant commutator (19) vanishes for space-like separations, which is consistent with causality. Using the form of Δ in (15), we can determine that

$$\rho = \frac{m\sqrt{z^2}}{2J_1(m\sqrt{z^2})}\, \theta(p_0)\delta(p^2 - m^2) \approx \theta(p_0)\delta(p^2 - m^2), \tag{20}$$

where the second equality holds near the light-cone with $m^2 = \frac{4}{(z^{\mathrm{T}})^2} > 0$. This can be compared with (18). We see that near the light-cone, where the discontinuity of $\tilde{n}^\mu \tilde{n}^\nu D_{\mu\nu}^{(\mathrm{PI})\ (\mathrm{R})}$ occurs, the spectral function is positive as it should be. (We emphasize here that the spectral function in (20) is not the inverse Fourier sine transform of $\left(\delta(z^2) - \frac{1}{(z^{\mathrm{T}})^2}\, \theta(z^2)\right)$; the latter, in fact, is an ill-behaved function which is ambiguous and regularization dependent.)

We note that all our results are manifestly invariant under translations. However, a coordinate dependent mass is quite unexpected. In fact, the spectral function ρ is normally a function of the momentum variables whereas in the present case, it depends on the transverse coordinates (in a translational invariant manner) as well. The normal assumptions in the derivation of the spectral representation include invariance of the vacuum under translations, namely,

$$e^{-iP\cdot x}|0\rangle = |0\rangle, \tag{21}$$

and our result suggests that this assumption in the present case may be violated for the $\tilde{n} \cdot A$ modes.

IV. DISCUSSION

As we have seen, the logarithmic term in the propagator leads, through its $\theta(x^2)$ discontinuity, to a coordinate dependent mass which would necessitate a redefinition of the vacuum. This behavior is highly unreasonable considering that we are dealing with a free theory. On the other hand, we have shown that the growth for large $\tilde{n} \cdot x$ of this logarithm is a consequence of a residual gauge symmetry (with parameter $\omega(x^{\mathrm{T}})$) present in the theory. From our earlier results [2], it follows that if one would fix this residual symmetry by imposing an appropriate extra gauge condition, this will lead to a propagator which is well behaved at infinity. For example, one could improve the behavior at infinity by adding to the effective action an extra gauge fixing term of the form

$$S_{\mathrm{extra}} = \lim_{\eta \to 0}\ -\frac{1}{2\eta} \int \mathrm{d}^4 x\, \delta(\tilde{n} \cdot x - \tau)(\partial \cdot A)^2, \tag{22}$$

which is defined at some fixed value of $\tilde{n} \cdot x = \tau$. This may then remove from the propagator the logarithmic term together with its $\theta(x^2)$ discontinuity We have not worked out the complete expression for such a gauge-fixed propagator, whose form is rather complicated and beyond the scope of this brief report. But we expect that the above behavior would also remove from the theory the unphysical coordinate dependent mass. One may understand this feature in a simple way by considering the classical equations of motion. If we decompose the vector potential as in (4), it is easy to check that the Maxwell equation

$$\partial_\mu F^{\mu\nu} = \left(\eta^{\mu\nu} \Box - \partial^\mu \partial^\nu\right) A_\nu = 0, \tag{23}$$

leads, in the light-cone gauge, to the component equations

$$\begin{aligned} \Box\, (\tilde{n} \cdot A) &= \tilde{n} \cdot \partial\, (\partial \cdot A)\,, \\ \Box A_\mu^{\mathrm{T}} &= \partial_\mu^{\mathrm{T}}\, (\partial \cdot A)\,. \end{aligned} \tag{24}$$

At first sight, it would seem that these do not yield massless equations for the components of the gauge field. However, by choosing an appropriate gauge parameter $\omega(n\cdot x, x^{\mathrm{T}})$ such that $\partial\cdot A = 0$ at some fixed $\tilde{n}\cdot x$, the extra terms in the above equations can be eliminated.

In summary, we have shown that, at the tree level, the propagator in the light-cone gauge with the Mandelstam prescription has a logarithmic growth for large values of $\tilde{n}\cdot x$ which is a consequence of a residual gauge symmetry. In turn, this induces an unphysical coordinate dependent mass for the $\tilde{n}\cdot A$ component of the gauge field. In order to remove this feature from Green's functions, one must eliminate the unphysical degrees of freedom by fixing consistently the residual gauge symmetry, although in the calculation of physical S-matrix elements, the Mandelstam-Leibbrandt prescription seems to be sufficient to obtain the correct behavior of physical quantities.

Acknowledgment:

We would like to thank Prof. J. C. Taylor for helpful discussions. This work was supported in part by the US DOE Grant number DE-FG 02-91ER40685 and by CNPq as well as by FAPESP, Brazil.

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